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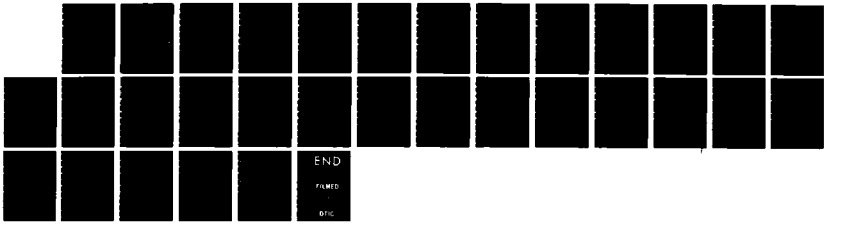

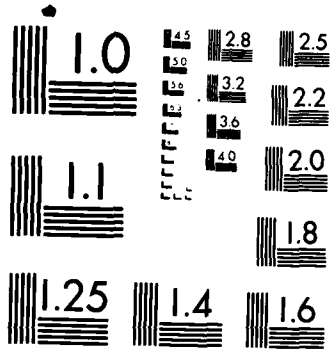

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Transient Solution of Acyclic Markov Chains

Raymond Marie
35042 Rennes Cedex, France

Andrew Reibman and Kishor Trivedi
Duke University
Durham, NC 27706 USA

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Transient Solution of Acyclic Markov Chains[†]

Raymond Marie[‡]
IRISA/Universite de Rennes
Campus de Beaulieu
35042 Rennes Cedex, France

Andrew Reibman and Kishor Trivedi
Department of Computer Science
Duke University
Durham, NC 27706 USA

Abstract

Continuous time Markov chains are commonly used in system reliability modeling. Increasing system complexity and non-Markovian behavior can drastically increase the size of a Markov model's state space. Special approximation techniques and numerical methods have been introduced to reduce the resources needed to solve Markov chain models. In this paper we discuss a method for automatically deriving exact transient solutions of Markov chains. The solutions derived are symbolic in t. Our approach can also provide solutions that are symbolic in other parameters. We extend our method to include parametric sensitivity analysis of the transient solution, and to provide cumulative measures of Markov chain behavior. We present three examples, one to show the use of our method in evaluating approximate solution techniques, one showing parametric sensitivity analysis of a large Markov model, and one demonstrating the computation of cumulative measures for an acyclic Markov reward processes.

Keywords: Markov Chains, Reliability Modeling, Sensitivity Analysis,
Symbolic Solution, Transient Analysis

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1. Introduction

Continuous time Markov chains (CTMC) are commonly used tools in computer systems modeling. CTMC have been used to model program behavior,[1] system performance,[2, 3] system reliability,[4, 5, 6] and system availability,[7, 8] and also in the combined evaluation of performance and reliability.[9, 10, 11] Although the limitation of exponentially distributed state occupancy times, as implied by a homogeneous CTMC, appears to be restrictive, it is possible to use the Coxian method of stages to allow arbitrary phase type distributions. [3, 6, 12, 13]

Much work done on Markov chain solution techniques has been motivated by queueing theory. Consequently, these efforts have concentrated on finding the "steady-state" solution of a Markov chain. In contrast, reliability modeling, for example, is interested primarily in transient solutions of Markov chains. Once a Markov chain model of a system has been constructed, several methods for finding its transient solution are available. Table 1 lists Markov chain transient solution methods and reliability modeling packages that employ them. A more detailed discussion of these reliability modeling packages can be found in the literature.[5] We briefly summarize the various methods here and provide appropriate references.

The Markov model of a system can be solved using integral equations, formally taking the convolution of the probability of entering the state with the probability of remaining in it.[14, 15] Alternatively, the Markov chain can be converted to a coupled set of homogeneous differential equations.[3] This set of equations can be solved using either numerical techniques or Laplace transforms. The third fundamental approach to transient solution is to convert the transition rate matrix to a form from which the solution can be easily obtained. Uniformization (or Randomization) is a transformation method that solves a CTMC by using a related discrete time Markov chain (DTMC).[14, 16] Other solution methods based on these three basic approaches include approximation techniques for evaluating large models,[17, 18] and the closed form exact technique that we present in this paper. Before we discuss our closed form solution technique, we will give an example of a typical acyclic Markov model.

Our example comes from a reliability model of an avionics computer system.[19] Figure 1 depicts the transition diagram of a Markov chain representing a TMR parallel redundant system with two cold

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spares. The individual components have lifetimes that are independent and exponentially distributed with parameter λ. The cold spares are assumed not to fail. When an individual component fails, a reconfiguration process with rate parameter δ begins. This process is guaranteed to reconfigure the system as long as a second fault does not occur before the reconfiguration is completed. The reliability of the system at time t, denoted $R(t)$, is 1-P {process is in a non-functioning state at time t}. So for the system in Figure 1, $R(t)$ is given by 1 - P {process in state A at time t}.

We will present a method for the automatic derivation of state probabilities of an acyclic Markov chain using closed form expressions that are symbolic in t. This approach can be extended to solve the CTMC symbolically in terms of other parameters. Although this computation might be done using an existing symbolic mathematics package, a package developed specifically for Markov chain solution is simpler, faster, and can be easily interfaced with existing performance modeling tools. Exact symbolic results (that previously could be obtained only by hand) give greater insight into actual system behavior by allowing us to easily study the relationship between input parameters and the resulting state probability distributions. Exact symbolic solutions also allow us to use Markov models as the basis for design optimization.[20] The derivation of our approach is based on the use of integral equations. In some ways, it is computationally similar to using symbolic Laplace transforms to solve systems of coupled ODE's. The program implementing our algorithm is called ACE (*Acyclic Markov Chain Evaluator*). We omit the discussion of the solution of cyclic chains in this paper. The analysis of a n-state single-entry single exit cycle (a simple case) requires the computation of the complex roots of a polynomial of degree $n+1$. In general, this is computationally expensive and probably not feasible for purely symbolic solution.

In section 2, we describe the mathematical basis for our method that is partially inspired by the program SPADE.[21] We then discuss the computation of "instantaneous measures" of CTMC behavior and describe the implementation of our procedure. In section 3, we discuss parametric sensitivity analysis of instantaneous measures. "Cumulative measures" of CTMC behavior are outlined in section 4. Three acyclic CTMC are analyzed in section 5, demonstrating the use of the various measures and

computational techniques previously described.

2. State Probability Computation and Instantaneous Measures

2.1 Basic Approach

Consider an acyclic continuous time Markov Chain. Let the states be numbered 1,2,...,N. $\mathbf{Q} = \{q_{ij}\}$ is the infinitesimal generator matrix of the CTMC. The transition rate from state i to state j is denoted by q_{ij}, and is restricted to be non-negative. We let $q_i = \sum_j q_{ij}$ denote the total exit rate from state i.

For any state i of an acyclic Markov chain, let $P_i(t)$ be the probability that the system is in state i at time t. For any state i, $P_i(t)$ may be written as a polynomial of the form

$$P_i(t) = \sum_j e^{\gamma_{ij} t} \left[\sum_k a_{ijk} t^k \right]. \tag{1}$$

The fact that the state probability distributions are of this form is easily derived. First observe that the initial state probabilities have this form. If we condition on the time of transition into (any non-initial) state i, the probability of being in that state at time t can be written as:

$$P_i(t) = \int_0^t \sum_{j \in J(i)} P_j(x) q_{ji} e^{-q_i(t-x)} dx \tag{2}$$

where $J(i)$ is the set of states with a transition leading into state i. By induction, it is easy to show that, if every $P_j(x)$ has the form (1), then so does any $P_i(x)$ derived using (2).

We now derive the equations needed to calculate the constants of equation (1) for any state of an acyclic Markov chain. Let $\Gamma(i)$ be the set of poles of the Laplace-Stieltjes Transform of $P_i(t)$, i.e. the set of the γ_{ij}'s of (1). We rename $\gamma^* \triangleq -q_i$ and define $\Gamma(J(i)) \triangleq \bigcup_{j \in J(i)} \Gamma(j)$. Setting $L(j) \triangleq |\Gamma(j)|$, we can write $\Gamma(j) = \{\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{jL(j)}\}$. With this numbering of the poles, $P_j(t)$ may be written as:

$$P_j(t) = \sum_{l=1}^{L(j)} e^{\gamma_{jl} t} \left[\sum_{k=0}^{K(j,l)} a_{jlk} t^k \right] \tag{3}$$

where $K(j,l)$ is the maximum power of t associated with pole γ_{jl} in $P_j(t)$ such that the formal

expression of a_{jlk} is different from zero. If $L(j) > 1$, it is easy to show using an inductive proof that, for any pole γ_{jl}, if the symbolic expression for $a_{jlK_{(k,l)}}$ is formally different from zero then for all values of $k < K(j,l)$, a_{jlk} is also formally different than zero.

If $L(j)=1$, then $P_j(t)$ is of the form $at^k e^{\gamma t}$. This corresponds to the case where there is only one directed path from the original state to state j and all the transition rates along this path are all equal, i.e. $q_j = -\gamma$. Only in the $L(j) = 1$ case is there a chance of having a symbolic expression formally equal to zero. Thus $P_i(t)$ may be written

$$P_i(t) = \sum_{j \in J(i)} \sum_{l=1}^{L(j)} \int_0^t e^{\gamma_{jl} x} \left[\sum_{k=0}^{K(j,l)} a_{jlk} x^k q_{ji} \right] e^{\gamma^*(t-x)} dx \ . \tag{4}$$

But, especially when the $P_j(t)$'s have common poles, i.e. when $|\Gamma(J(i))| < \sum_{j \in J(i)} |\Gamma(j)|$, this expression reduces to

$$P_i(t) = \sum_{\gamma \in \Gamma(J(i))} \int_0^t e^{\gamma x} \left[\sum_{j \in J(i)} \sum_{l=1}^{L(j)} 1_{(\gamma_{jl} = \gamma)} \left(\sum_{k=0}^{K(j,l)} q_{ji} a_{jlk} x^k \right) \right] e^{\gamma^*(t-x)} dx$$

$$= \sum_{\gamma \in \Gamma(J(i))} \int_0^t \sum_{k=0}^{K_i(\gamma)} \tilde{a}_{\gamma,k} x^k e^{\gamma x} e^{\gamma^*(t-x)} dx \tag{5}$$

where

$$\tilde{a}_{\gamma,k} \triangleq \sum_{j \in J(i)} \sum_{l=1}^{L(j)} 1_{(\gamma_{jl} = \gamma)} q_{ji} a_{jlk}, \qquad \gamma \in \Gamma(J(i)) \tag{6}$$

and

$$K_i(\gamma) \triangleq \max_{j \in J(i)} \left\{ K(j,r) \mid r = 1,\ldots,L(j), \text{ and } \gamma_{jr} = \gamma \right\}. \tag{7}$$

Moving the integral inside the summation we obtain

$$P_i(t) = \sum_{\gamma \in \Gamma(J(i))} \sum_{k=0}^{K_i(\gamma)} e^{\gamma^* t} \tilde{a}_{\gamma,k} \int_0^t x^k e^{(\gamma - \gamma^*)x} dx \ . \tag{8}$$

The resolution of this integral depends on whether γ differs from γ^*. If $\gamma = \gamma^*$, then,

$$\int_0^t x^k e^{(\gamma - \gamma^*)x} dx = \int_0^t x^k dx = \frac{t^{k+1}}{(k+1)} \ . \tag{9}$$

Otherwise, when $\alpha = (\gamma - \gamma^*)$, integration by parts yields

$$\int_0^t x^k e^{\alpha x} dx = e^{\alpha t} \left[\sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \frac{t^{k-l}}{\alpha^{l+1}} \right] + (-1)^{k+1} \frac{k!}{\alpha^{k+1}} . \tag{10}$$

We can now write the equation for $P_i(t)$

$$P_i(t) = \sum_{\gamma \in \Gamma(J(i))} \left\{ 1_{(\gamma \neq \gamma^*)} \sum_{k=0}^{K_i(\gamma)} \left[e^{\gamma t} \tilde{a}_{\gamma,k} \left\{ \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \frac{t^{k-l}}{\alpha^{l+1}} \right\} \right. \right.$$

$$\left. \left. + \tilde{a}_{\gamma,k} (-1)^{k+1} \frac{k!}{\alpha^{k+1}} e^{\gamma \alpha} \right] + 1_{(\gamma = \gamma^*)} \sum_{k=0}^{K_i(\gamma^*)} e^{\gamma^* \alpha} \tilde{a}_{\gamma^*,k} \frac{t^{k+1}}{(k+1)} \right\} . \tag{11}$$

We note that

$$\Gamma(i) = \{\gamma^*\} \cup \Gamma(J(i)) \tag{12}$$

and that

$$K(i,l) = \begin{cases} K_i(\gamma_u) & \text{if } \gamma_u \neq \gamma^* \\ K_i(\gamma_u)+1 & \text{if } \gamma_u = \gamma^* \text{ and } \gamma^* \in \Gamma(J(i)) \\ 0 & \text{if } \gamma_u = \gamma^* \text{ and } \gamma^* \notin \Gamma(J(i)) \end{cases} \quad , \quad l = 1,\ldots,L(i) \tag{13}$$

From (11) note that, if $u = k-l$, we have

$$\sum_{k=0}^{K_i(\gamma)} \tilde{a}_{\gamma,k} \left[\sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!} \times \frac{t^{k-l}}{\alpha^{l+1}} \right]$$

$$= \sum_{k=0}^{K_i(\gamma)} \tilde{a}_{\gamma,k} \left[\sum_{u=0}^{k} (-1)^{k-u} \frac{k!}{u!} \times \frac{t^u}{\alpha^{k+1-u}} \right]$$

$$= \sum_{u=0}^{K_i(\gamma)} t^u \sum_{k=u}^{K_i(\gamma)} \tilde{a}_{\gamma,k} (-1)^{k-u} \frac{k!}{u!} \times \frac{1}{\alpha^{k+1-u}}$$

$$= \sum_{k=0}^{K_i(\gamma)} t^k \sum_{r=k}^{K_i(\gamma)} \tilde{a}_{\gamma,r} (-1)^{r-k} \frac{r!}{k!} \times \frac{1}{\alpha^{r+1-k}} .$$

If we write $P_i(t)$ as

$$P_i(t) = \sum_{l=1}^{L(i)} e^{\gamma_l t} \sum_{k=0}^{K(i,l)} a'_{\gamma_l,k} t^k , \tag{14}$$

for all $\gamma \neq \gamma^*$, we have

$$a'_{\gamma,k} = \sum_{r=k}^{K_i(\gamma)} (-1)^{r-k} \tilde{a}_{\gamma,r} \frac{r!}{k!} \frac{1}{(\gamma-\gamma^*)^{r-k+1}} \quad , \quad k = 0,\ldots,K_i(\gamma) \tag{15}$$

and for γ^*

$$a'_{\gamma^*,0} = \sum_{l=1}^{L(i)} 1_{(\gamma_{il} \neq \gamma^*)} \sum_{k=0}^{K_i(\gamma_l)} (-1)^{k+1} \cdot k! \frac{\tilde{a}_{\gamma_{il},k}}{(\gamma_{il} - \gamma^*)^{k+1}}$$

$$= -\sum_{l=1}^{L(i)} 1_{(\gamma_{il} \neq \gamma^*)} a'_{\gamma_{il},0} \quad . \tag{16}$$

Moreover, if $\gamma^* \in \Gamma(J(i))$,

$$a'_{\gamma^*,k+1} = \frac{\tilde{a}_{\gamma^*,k}}{(k+1)} \qquad k = 0,\ldots,K_i(\gamma^*) \tag{17}$$

With these equations, we can easily compute the coefficients of the polynomials in t that multiply the exponentials in the state probability expressions.

If we associate with each state i a reward level r_i, then the exp ted instantaneous reward rate for the process at time t is given by $\sum_i r_i P_i(t)$. Its polynomial expression can be easily obtained from the polynomial expressions of the "$P_i(t)$"'s. If we consider the system to be up in all non-absorbing states, then we assign $r_i = 1$ to all such states and $r_i = 0$ to all absorbing states. In such a case, the evaluation of $\sum r_i P_i(t)$ will give the system reliability $R(t)$. On the other hand, if some of the non-absorbing states are system down states which are assigned reward 0, then the evaluation of $\sum r_i P_i(t)$ will give the point (instantaneous) availability $A(t)$ of the system. In the general case, we are not restricted to binary values of the reward rate and we can allow several (degraded) levels of functionality.

2.2 Implementation

In this section we discuss the procedure used by the ACE program to compute state probability expressions for an acyclic Markov Chain. We first note that equation (15) can be re-written in a recursive fashion to greatly reduce the number of operations needed to compute the coefficients of the state probability expressions. In place of equation (15), we can write

$$a'_{\gamma,K_i(\gamma)} = \frac{\tilde{a}_{\gamma,K_i(\gamma)}}{\alpha} \tag{18}$$

and

$$a'_{\gamma,k} = \frac{(\tilde{a}_{\gamma,k} - (k+1)\, a'_{\gamma,k+1})}{\alpha} \qquad k = K_i(\gamma)-1,..,0 \tag{19}$$

The equations for computing the coefficients of the outgoing poles (16 and 17) remain unchanged.

When the procedure begins, the states are sorted according to the partial order induced by the transitions. For state i, the probability expression is computed by first determining all the poles of states that have transitions leading to state i. Coefficients for the polynomials multiplying the incoming exponential terms are computed using equations (18) and (19). If the pole associated with state i's outgoing transition rate is not in the incoming set, the new pole's polynomial multiplier is a constant computed using equation (16). If the outgoing pole is also in the set of incoming poles, the degree of its polynomial will be incremented. The new coefficients for the incremented polynomial can be computed using equation (16) and (17).

We comment here on the execution speed of this procedure. We first observe that, in a partially ordered chain, state i has at most $i-1$ ancestors. The number of coefficients in a state probability expression is bounded above by the number of ancestors the state has. Therefore, to compute the state probability distributions of all the states in an n-state acyclic Markov chain we need to compute at most $\frac{(n+1)*n}{2}$ coefficients. The procedure described above requires at most 2 floating point multiplies and/or divides per coefficient computed. Thus the number of multiplications required to compute all the state probability expressions for an acyclic chain is bounded by $(n+1)*n$ multiplies. The number of floating point additions for state i is bounded by $i-1$. So even if additions are included in the operations count, the total number of floating point operations is bounded by $1.5\ n*(n+1)$. In contrast, a typical ODE solution package will require about $20*n^3$ floating multiplies to compute that transient solution of an arbitrary CTMC for a single point (or small set of grid points).[22] Other methods that do not take into account the special structure of acyclic CTMC have comparable runtimes. Even if other numerical methods are adapted to specifically solve acyclic chains, the ACE algorithm is still competitive.

The first version of ACE supports an unlimited number of symbolic variables as input but generates answers that are symbolic only in t. The coefficients of the polynomials in t that multiply the exponentials are numeric. The second version of ACE, which is currently being implemented, will be fully symbolic in one variable and numeric in other variables, i.e. the coefficients of the polynomials in t that multiply the exponentials have both numeric and symbolic parts. This allows us to conduct a parametric sensitivity analysis in a fully symbolic fashion, without repeated computation.

3. Sensitivity Analysis

It is often of interest to the design engineer to determine the performance or reliability "bottleneck" of the system.[8, 23] Towards this end, it is desirable to evaluate the derivative of the desired measure with respect to various significant system parameters. The parameter with the largest derivative deserves the designer's interest in the quest to improve the characteristics of the designed system. Such derivatives could also be used in an overall system optimization effort based on gradient search techniques. We discuss such automated sensitivity analysis next.

3.1 Basic Approach

We assume that some transition rates q_{ij} are functions of some parameter λ. We seek a symbolic expression in t for the derivative $\frac{dP_i(t)}{d\lambda}$ for a given value of λ.

Let:

$\mathbf{P}$ be the row vector of the $P_i(t)$'s,

$\mathbf{Q}$ be the previously defined triangular infinitesimal generator matrix of the acyclic CTMC,

$\mathbf{S}$ be the row vector of the sensitivities $\frac{dP_i(t)}{d\lambda}$,

$\dot{\mathbf{P}}$(respectively $\dot{\mathbf{S}}$) be the vector of the derivative with respect to time t of $\mathbf{P}$(respectively $\mathbf{S}$).

Then, from

$$\dot{\mathbf{P}} = \mathbf{PQ}\,, \tag{20}$$

we obtain

$$\dot{\mathbf{S}} = \mathbf{SQ} + \mathbf{PV} \tag{21}$$

where matrix $\mathbf{V}$ is the derivative of $\mathbf{Q}$ with respect to λ, i.e. $\mathbf{V} \triangleq \{v_{ij}\}$ such that $v_{ij} \triangleq \frac{dq_{ij}}{d\lambda}$. So $\mathbf{S}$ is the solution of the first order differential system (21).

Defining $\mathcal{J}(i)=\{j \mid 1 \le j \le N\ ,\ j \neq i\ ,$ and $v_{ji} \neq 0\}$ and taking into account the triangularity of $\mathbf{Q}$, we have

$$\dot{S}_i(t) = \sum_{j \in \mathcal{J}(i)} q_{ji} S_j(t) + q_{ii} S_i(t) + \sum_{j \in \mathcal{J}(i)} v_{ji} P_j(t) + v_{ii} P_i(t) \tag{22}$$

or

$$S_i(t) - q_{ii}\, S_i(t) = W_i(t) \tag{23}$$

where

$$W_i(t) \triangleq \sum_{j \in J(i)} q_{ji}\, S_j(t) + \sum_{j \in \underline{J}(i)} v_{ji}\, P_j(t) + v_{ii}\, P_i(t)\,. \tag{24}$$

Its solution is of the form [24]

$$S_i(t) = e^{-q_{ii}t} \int_0^t W_i(x) e^{q_{ii}x}\, dx \;+\; C_i\, e^{-q_{ii}t}\,. \tag{25}$$

Because because for all i , $P_i(0)$ is a constant, we have $C_i = 0$ for all i. Using the notation γ^* for $-q_{ii}$, we then get

$$S_i(t) = \sum_{j \in J(i)} q_{ji} \int_0^t S_j(x) e^{\gamma^*(t-x)} dx \;+\; \sum_{j \in \underline{J}(i)} v_{ji} \int_0^t P_j(x) e^{\gamma^*(t-x)} dx \;+\; v_{ii} \int_0^t P_i(x) e^{\gamma^*(t-x)} dx\,. \tag{26}$$

In initial state i, $J(i) = \underline{J}(i) = \phi$ and $S_i(t)$ has the following form:

$$S_i(t) = v_{ii} \int_0^t P_{(x)} e^{\gamma^*(t-x)} dx$$

$$= v_{ii}\, te^{-q_{ii}t}\,. \tag{27}$$

It is easily seen that, for any state i, i=1,..,N, $S_i(t)$ may be written as a polynomial of the form

$$S_i(t) = \sum_j e^{\gamma_{ij}t} \left[\sum_k b_{ijk}\, t^k \right] \tag{28}$$

Moreover, the set of poles corresponding to $S_i(t)$ is included in $\Gamma(i)$.

To get the recursive algorithm for computing the polynomial form of $S_i(t)$, we first assume that

$$S_j(t) = \sum_{l=1}^{L(j)} e^{\gamma_{jl}t} \left[\sum_{k=0}^{\overline{K}(j,l)} b_{jlk}\, t^k \right] \quad , \quad j \in \underline{J}(i) \tag{29}$$

where, by convention $\overline{K}(j,l) = -1$ if there is no exponential term with respect to γ_{jl} while $\gamma_{jl} \in \Gamma(j)$. Then, using equation (26), we show that $S_i(t)$ may be also written in the same form and we determine the expression of the new $\overline{K}(i,l)$'s and b_{ilk}'s:

$$S_i(t) = \sum_{j \in J(i)} q_{ji} \sum_{l=1}^{L(j)} \int_0^t e^{\gamma_{jl}x} \left[\sum_{k=0}^{\overline{K}(j,l)} b_{jlk}\, x^k \right] e^{\gamma^*(t-x)} dx$$

$$+ \sum_{j \in \underline{J}(i)} v_{ji} \sum_{l=1}^{L(j)} \int_0^t e^{\gamma_{jl}x} \left[\sum_{k=0}^{K(j,l)} a_{jlk}\, x^k \right] e^{\gamma^*(t-x)} dx$$

$$+v_{ii}\sum_{l=1}^{L(i)}\int_0^t e^{\gamma_{il}x}\left[\sum_{k=0}^{K(i,l)}a_{ilk}\,x^k\right]e^{\gamma^*(t-x)}dx \quad . \tag{30}$$

Let us introduce the following definitions

$$\overline{K}_i(\gamma)\triangleq\max_{j\in J(i)}\left\{\overline{K}(j,r)\mid r=1,..L(j)\text{ and }\gamma_{jr}=\gamma\right\} \tag{31a}$$

$$\underline{K}_i(\gamma)\triangleq\max_{j\in J(i)}\left\{K(j,r)\mid r=1,..L(j)\text{ and }\gamma_{jr}=\gamma\right\} \quad . \tag{31b}$$

By convention, we take $\overline{K}_i(\gamma)=-1$, (respectively $\underline{K}_i(\gamma)=-1$), if the set in the definition (31a) (respectively (31b)) is empty. Then from relation (30) and taking into account the influence of the convolution operation over the power of t, we get:

For $\gamma\neq\gamma^*$

$$\overline{K}(i,l)\triangleq\max\left\{\overline{K}_i(\gamma)\,,\,\underline{K}_i(\gamma)\,,\,(-1_{(v_{ii}=0)}+1_{(v_{ii}\neq 0)}K(i,l))\right\} \quad . \tag{32}$$

Note that $\overline{K}(i,\gamma)=-1$ iff $v_{ii}=0$ and $\overline{K}_i(\gamma)=\underline{K}_i(\gamma)=-1$.

For γ^*

$$\overline{K}(i,\gamma^*)\triangleq\max\left\{\overline{K}_i(\gamma^*)+1_{(\overline{K}_i(\gamma^*)\geq 0)}\,,\,\underline{K}_i(\gamma)+1_{(\underline{K}_i(\gamma^*)\geq 0)}\,,\,(-1_{(v_{ii}=0)}+1_{(v_{ii}\neq 0)}(K(i\gamma^*)+1))\right\} \quad . \tag{33}$$

On the other hand, we also get from (30)

$$S_i(t)=\sum_{\gamma\in\Gamma(J(i)}\sum_{k=0}^{\overline{K}_i(\gamma)}e^{\gamma^* t}\,\tilde{b}_{\gamma,k}\int_0^t x^k\,e^{(\gamma-\gamma^*)}dx \tag{34}$$

where

$$\begin{aligned}\tilde{b}_{\gamma,k}\triangleq\;&1_{(\overline{K}_i(\gamma)\geq 0)}\sum_{j\in J(i)}q_{ji}\sum_{l=1}^{L(j)}1_{(\gamma_{jl}=\gamma)}b_{jlk}\\&+1_{(\underline{K}_i(\gamma)\geq 0)}\sum_{j\in J(i)}v_{ji}\sum_{l=1}^{L(j)}1_{(\gamma_{jl}=\gamma)}a_{jlk}\\&+v_{ii}\,a'_{\gamma,k}\qquad,\qquad\forall k=0,...\tilde{K}_i(\gamma)\,,\,\forall\gamma\in\Gamma(i)\end{aligned} \tag{35}$$

and

$$\tilde{K}_i(\gamma)\triangleq\begin{cases}\overline{K}(i,\gamma) & \text{if }\gamma\neq\gamma^*\\ \overline{K}(i,\gamma^*)-1 & \text{otherwise}\end{cases} \tag{36}$$

Note that $\tilde{b}_{\gamma^*,0} = a'_{\gamma^*,0} v_{ii}$ if $\gamma^* \notin \Gamma(J(i))$. The form of the relation (34) is equivalent to the one of relation (5) up to the last term of (34). Therefore we get similar recursive relations for the computation of the b coefficients.

For $\gamma \neq \gamma^*$

$$b'_{\gamma,\bar{K}(i,\gamma)} = \frac{\tilde{b}_{\gamma,\bar{K}(i,\gamma)}}{\alpha} \tag{37a}$$

$$b'_{\gamma,k} = \frac{\left(\tilde{b}_{\gamma,k} - (k+1) b'_{\gamma,k+1}\right)}{\alpha} \qquad k = (\bar{K}(i,\gamma)-1),\ldots,0 \,. \tag{37b}$$

For γ^*

$$b'_{\gamma^*,0} = -\sum_{l=1}^{L(i)} 1_{(\gamma_{il} \neq \gamma^* \text{ and } \bar{K}(i,\gamma) \geq 0)} \, b'_{\gamma_{il},0} \tag{38a}$$

$$b'_{\gamma^*,k} = \frac{\tilde{b}_{\gamma^*,k-1}}{k} \qquad k = 1,\ldots,\bar{K}(i,\gamma^*) \,. \tag{38b}$$

Note that if there are no poles other than γ^*, the term $b'_{\gamma^*,0}$ does not exit. Note also, that the number of floating point operations is the approximately the same as the number used in the computation of the state probabilities.

4. Cumulative Measures

Thus far, we have discussed instantaneous measures associated with a CTMC. There are many cumulative measures of interest as well. The most common such measure is the mean time to failure, We first discuss cumulative measures for a Markov reward process and then specialize to various cases.[14] Let $Z(t)$ be the state of the CTMC at time t. Define the accumulated reward in the interval $[0,t]$ as[9, 11]

$$Y(t) = \int_0^t r_{Z(\tau)} d\tau \,. \tag{39}$$

Then the expected cumulative reward in the interva. [0,t] is given by:

$$E[Y(t)] = \sum_{i=1}^{N} \int_0^t r_i P_i(x) dx \tag{40}$$

If

$$\tilde{c}_{\gamma,k} \triangleq \sum_{i=1}^{N} r_i \sum_{l=1}^{L(i)} 1_{(\gamma_{il}=\gamma \text{ and } k \le K(i,l))} a_{ilk} \tag{41}$$

and

$$K(\gamma) \triangleq \text{maximum power of } t \text{ with respect to pole } \gamma \text{ over all states with } r_i \neq 0$$

then

$$E[Y(t)] = \sum_{\gamma} \sum_{k=0}^{K(\gamma)} \tilde{c}_{\gamma,k} \int_0^t x^k e^{\gamma x} dx \tag{42}$$

Note that a pole $\gamma=0$ had been introduced by all absorbing states and consequently

$$\tilde{c}_{0,k} = 0 \qquad \text{for } k > 0 .$$

Moreover, if the reward is null for all the absorbing states, $\tilde{c}_{0,0} = 0$. After the integration we get

$$E[Y(t)] = \sum_{\gamma \neq 0} \sum_{k=0}^{K(\gamma)} c_{\gamma,k} t^k e^{\gamma t} + c_{0,0} + c_{0,1} t \tag{43}$$

where

$$c_{\gamma,K(\gamma)} = \frac{\tilde{c}_{\gamma,K(\gamma)}}{\gamma} \tag{44a}$$

$$c_{\gamma,k} = \frac{(\tilde{c}_{\gamma,k} - (k+1)\, c_{\gamma,k+1})}{\gamma} \qquad k=0,..,K(\gamma)-1 \ , \ \forall \gamma \neq 0 \tag{44b}$$

and

$$c_{0,0} = -\sum_{\gamma \neq 0} c_{\gamma,0} \qquad c_{0,1} = \tilde{c}_{0,0} . \tag{44c}$$

We have the following special cases:

If $r_i = \begin{cases} 0 & \text{if } i \text{ is an absorbing state} \\ 1 & \text{otherwise} \end{cases}$

Then:

- $\lim_{t \to \infty} E[Y(t)] = c_{0,0}$ is equivalent to the "mean time to absorption" (MTTA).

  Note that in this case $c_{0,1} = 0$

If $r_i = \begin{cases} 0 & \text{if } i \in \text{set of } \textit{down} \text{ states} \\ 1 & \text{if } i \in \text{set of } \textit{up} \text{ states} \end{cases}$

where the absorbing states must belong to the set of *down* states.

Then:

- $\lim_{t \to \infty} E[Y(t)]$ is equivalent to the "mean up time before absorption" (MUTBA).

- $E[Y(t)]$ is the "expected up time" on the interval $[0,t]$.
- $\frac{E[Y(t)]}{t}$ is the "expected interval availability" on $[0,t]$.

The sensitivity of $E[Y(t)]$ can also be derived. If we define Λ to be the set of numerical values of λ which would make two formally different poles numerically identical, then we can write

$$\frac{dE[(Y(t)]}{d\lambda} = \sum_{i=1}^{N} r_i \int_0^t \frac{dP_i(z)}{d\lambda} dx \qquad \forall \lambda \notin \Lambda$$

$$= \sum_{i=1}^{N} r_i \int_0^t S_i(x)\,dx \tag{45}$$

Here the differentiation under the integral may be taken because, for all $\lambda \notin \Lambda$, the integral exists and the derivative with respect to λ is continuous.[25]

Let $\overline{K}(\gamma) \triangleq$ the maximum power of t with respect to pole γ in the $S_i(t)$ expressions over all states i such that $r_i \neq 0$. Let

$$\tilde{d}_{\gamma,k} \triangleq \sum_{i=1}^{N} r_i \sum_{l=1}^{L(i)} 1_{(\gamma_{il}=\gamma \text{ and } k \leq \overline{K}(i,l))}\, b_{ilk} \qquad k=0,..,\overline{K}(\gamma) \tag{46}$$

then

$$\frac{dE[Y(t)]}{dt} = \sum_{\gamma} \sum_{k=0}^{\overline{K}(\gamma)} \tilde{d}_{\gamma,k} \int_0^t x^k e^{\gamma x}\, dx \ . \tag{47}$$

The integration is quite similar to the one above of $E[Y(t)]$. We get

$$\frac{dE[Y(t)]}{d\lambda} = \sum_{\gamma \neq 0} \sum_{k=0}^{\overline{K}(\gamma)} d_{\gamma,k}\, t^k e^{\gamma t} + d_{0,0} + d_{0,1} t \tag{48}$$

where

$$d_{\gamma,\overline{K}(\gamma)} = \frac{\tilde{d}_{\gamma,\overline{K}(\gamma)}}{\gamma} \tag{49a}$$

$$d_{\gamma,k} = \frac{(\tilde{d}_{\gamma,k} - (k+1)\, d_{\gamma,k+1})}{\gamma} \qquad k = \overline{K}(\gamma)-1,..,0 \ , \ \forall \gamma \neq 0 \tag{49b}$$

and

$$d_{0,0} = -\sum_{\gamma \neq 0} d_{\gamma,0} \quad , \quad d_{0,1} = \tilde{d}_{0,0} \ . \tag{49c}$$

Note that, if the absorbing state is unique, or if the reward is null for all the absorbing states, then $\tilde{d}_{0,0} = 0$ and therefore $d_{0,1} = 0$.

5. Symbolic and Numerical Examples

In this section, we demonstrate the use of ACE state probability computation, sensitivity analysis, and expected reward computation. Our first example is the solution and sensitivity analysis of a medium-sized acyclic CTMC that is used in the analysis of a stochastic PERT network.[26] We also determine the sensitivity of the solution with respect to one of the transition rates.

Our second example considers the fault-tolerant computer system model given in Figure 1. First we solve for the reliability of the original model. We compare the exact solution to the reliability estimate obtained by using two approximate models. Assigning rewards to various states of the model, we then compute cumulative measures of the system's performance and analyze the sensitivity of these performance estimates with respect to the system parameters.

5.1 Solution and Sensitivity Analysis of a Stochastic PERT Network

We consider the analysis of a stochastic PERT network.[26] The chain that must be solved to analyze the network is given in Figure 2a. The chain has 24 states and its solution is clearly too complex for practical hand computation or a general purpose symbolic solver. In stochastic PERT analysis, we are interested in the distribution of the time needed to reach the absorbing state. For an absorbing state, the probability that the process reaches the state on or before time t is given by the probability of being in the state at time t. Using the parameter values given in Figure 2b, we use the numerical ACE algorithm to find the probability of being in the absorbing state (state 24) as a function of time t. The numerical expression in terms of t is given in Table 2.

In Table 3, we give the sensitivity of the solution with respect to parameter μ_2.

5.2 Analysis of Exact and Approximate Models of A TMR System

We consider the example presented in Figure 1, a TMR (Triple Modular Redundancy) System with two cold spares. We assume that the failure rate of an active unit is λ and that of a spare unit is zero. Upon occurrence of a failure, the system takes an exponentially distributed time (with parameter δ) to switch out the failed unit. Should another active unit fail during this switching process, we consider the system to have failed. We do not allow any repairs of failed units. The state diagram of this system is

shown in Figure 1.

We note that the reliability of the system at time t, denoted $R(t)$, is given by $1-P_A(t)$. Taking λ to be 10^{-4} and δ to be 1.0, we use the ACE package to derive the expression for $P_A(t)$ given in Table 4a.

We consider an approximate model of the system. The instantaneous coverage model, replaces the recovery states by probabilistic branch-points. The coverage constant c denotes the probability of successful recovery from a processor failure. In this example, recovery is unsuccessful only if a second processor failure occurs during the recovery period. Thus the probability of successful recovery can be computed as $c = \frac{\delta}{(2\lambda + \delta)}$.[18] A special case of this model, the perfect coverage model, assumes that recovery is always successful. In this case, $c = 1.0$. The aggregated chain for the approximate model is shown in Figure 3. The unreliability expressions for the instantaneous coverage and perfect coverage models are given in Table 4b and Table 4c. In Figure 4 we plot $-log_{10}$ of the system unreliability as a function of time. We see that the constant coverage model provides a fairly tight lower bound on the actual solution.[27] The perfect coverage model provides a less tight upper bound.

5.3 Computation of Cumulative Measures for a TMR System

We consider the computation of cumulative reward measures for the chain analyzed in the previous section. It is natural to assign reward rate one to states (3,2), (3,1), (3,0), and (2,0) and reward rate zero to the absorbing state A. If the switching time is small, we may assign reward rate one to states (2,2) and (2,1) if the system requirements allow for temporary "down time". On the other hand, we may assign reward rate 0 to these two states if we are interested in actual system "up time" in the given observation period.

Using the numeric values suggested previously, $\lambda = 10^{-4}$ and $\delta = 1.0$, we evaluate the TMR system's mean up time in the interval [0,t]. Using the reward vector with reward level 0 assigned to the recovery states, we give the expression for the expected cumulative reward earned up to time t in Table 5. We note that the constant term (14,996) in the expression gives expected up time before absorption (MUTBA). In Tables 6a and 6b respectively, we give the sensitivity of the expected reward with respect to λ and δ.

6. Conclusions

Efficient generation of exact numeric and exact symbolic expressions for the transient behavior of CTMC should provide at least three benefits. First, it should be possible to compare the results obtained by approximate solution methods for small to medium sized CTMC with exact solutions for the same problems. By showing the magnitude and nature of the error that approximate solutions introduce, this type of analysis should provide a good indication of an aggregation/approximation technique's utility for larger, more realistic problems. Second, symbolic solutions allow us to easily examine the influence of changing parameter values on the solutions of Markov models. This type of investigation could be expensive using conventional simulation or numerical solution techniques. Symbolic solutions also provide the basis for using Markov models in mathematical programs for design optimization. By providing exact and exact symbolic solutions of CTMC, the ACE package should enhance our ability to study Markov reliability models, and approximation techniques for their solution.

Further theoretical efforts include the efficient closed-form solution of cyclic chains, the analysis of restricted semi-markov processes, and reward based computations. Practical extensions include interfacing the ACE procedure with existing modeling tools and the addition of a block definition and solution facility. The user will be able to define blocks of states with fixed entry and exit points. The blocks could be evaluated by direct insertion of their states into the chain. Alternatively, the block could be solved in isolation and replaced by approximately equivalent acyclic sub-chains. This capability should further facilitate the use of ACE in evaluating aggregation methods.

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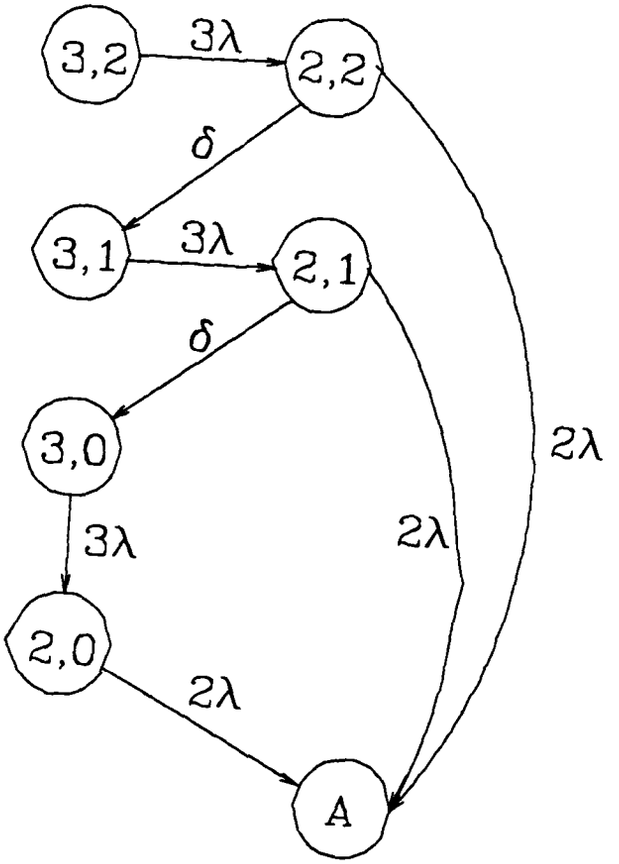

Figure 1 - CTMC for TMR System with Two Cold Spares

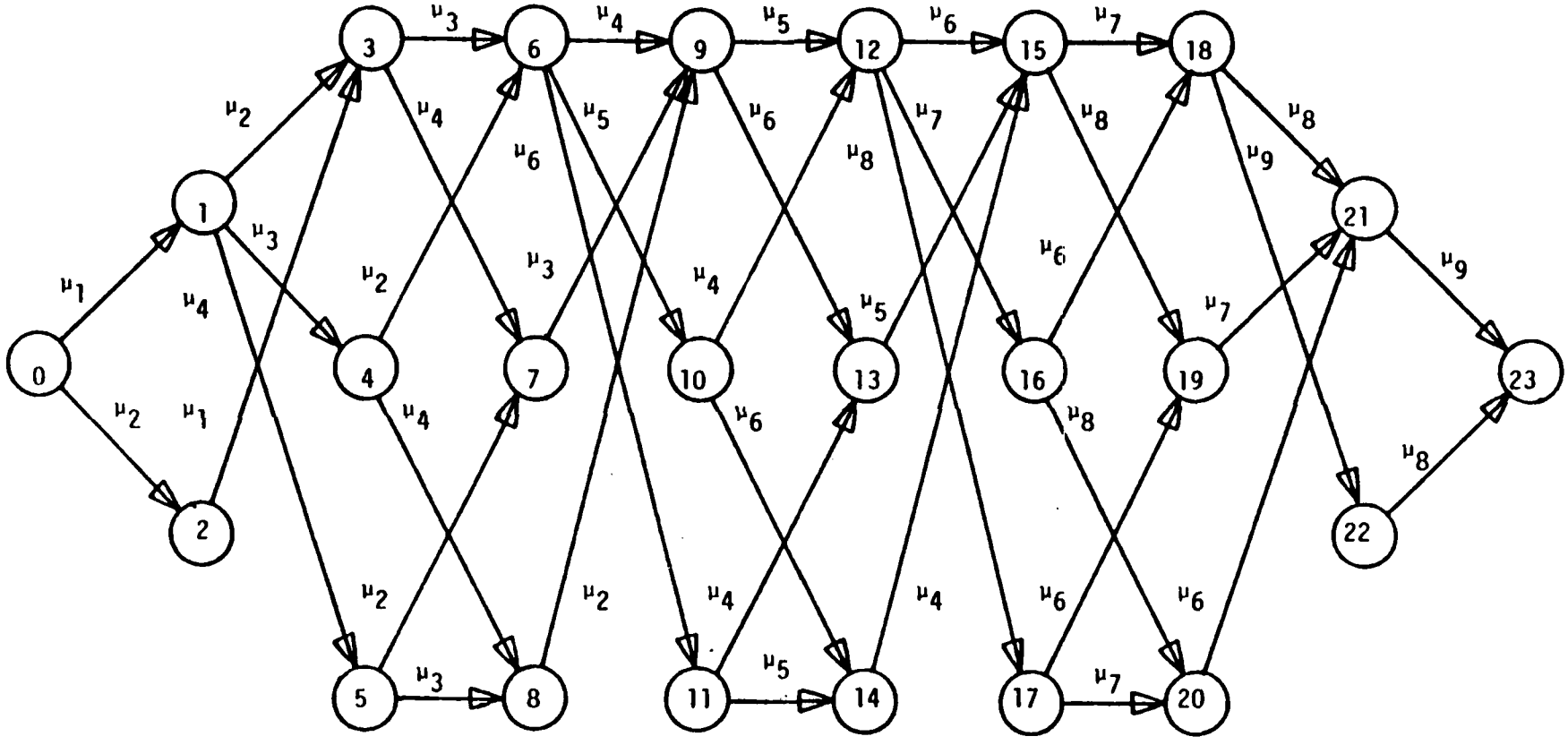

Figure 2 - CTMC for Stochastic PERT Network

$\mu = 1.5$, $\mu_2 = 1.2$, $\mu_3 = .6$, $\mu_4 = .5$, $\mu_5 = .8$, $\mu_6 = .5$, $\mu_7 = .6$, $\mu_8 = 1.5$, $\mu_9 = 2.0$

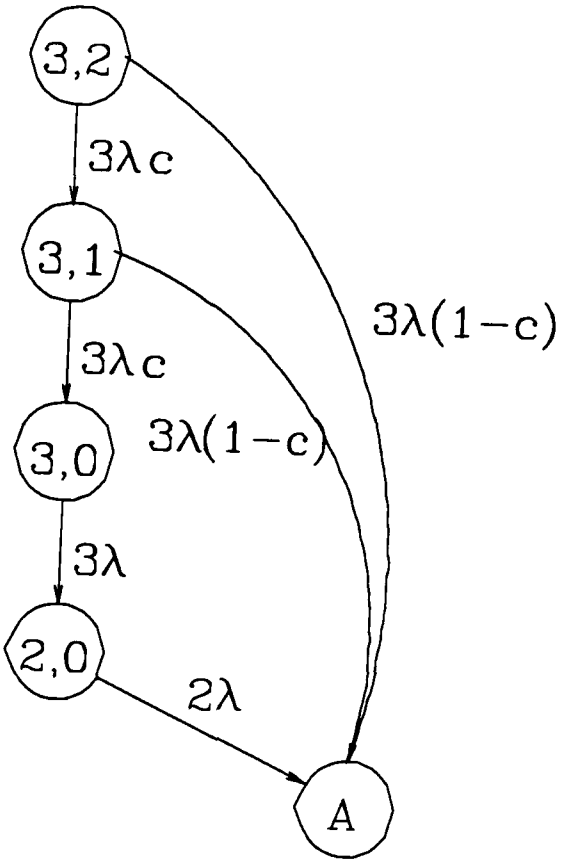

Figure 3 - Aggregated CTMC for TMR System with Two Cold Spares

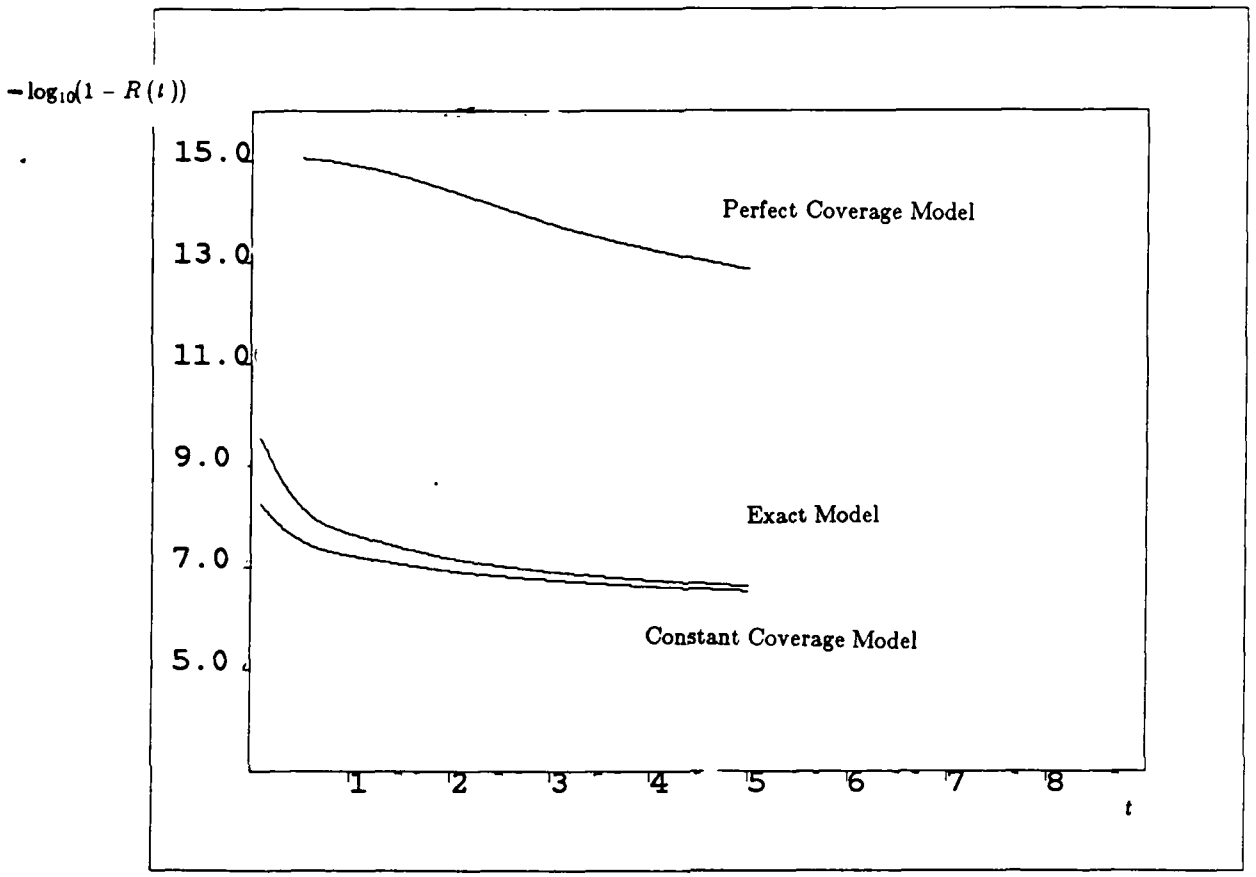

Figure 4 - Comparison of TMR Model Unreliability Estimates

Method: Simulation

Package: CAST [28]
Note: Allows full simulation of a restricted class of systems

Method: Differential Equations (Numerical Solution)

Package: HARP [18]
Domain: Homogeneous (cyclic) and (acyclic) Non-Homogeneous CTMC

Method: Differential Equations (Laplace Solution)

Package: SURF [6]
Domain: Non-Markov Processes
Note: Approximate solution using Coxian method of stages

Method: Integral Equations (Numerical Solution)

Package: Care III Coverage Model [15]
Domain: Semi-Markov Processes

Package: Care III Reliability Model [15]
Domain: Non-Homogeneous acyclic CTMC

Method: Uniformization

Package: SAVE [8]
Domain: Homogeneous Markov Processes with Rewards

Method: Eigenvalue Decomposition

Package: ARIES [29]
Domain: Cyclic Homogeneous CTMC
Note: Poles and their coefficients derived numerically

Method: Closed Form Solution

Package: ARIES [29]
Domain: Acyclic Homogeneous CTMC
Note: Distinct Eigenvalues Required

Package: ACE
Domain: Acyclic Homogeneous CTMC
Note: State probability expression derived symbolically in t
No restriction on eigenvalues
Other symbolic variables optionally available
Sensitivity, cumulative, and reward-based measures

Table 1 - Reliability Modeling Packages Employing Markov Chain Techniques

```
+ (-6.58467) * (t ** 0) * exp (-2.7 t)
+ (-0.604801) * (t ** 0) * exp (-3 t)
+ (-58.6153) * (t ** 0) * exp (-1.5 t)
+ (10.3929) * (t ** 1) * exp (-1.5 t)
+ (0.959701) * (t ** 0) * exp (-1.1 t)
+ (-44.0977) * (t ** 1) * exp (-1.1 t)
+ (412.962) * (t ** 0) * exp (-1.7 t)
+ (-279.099) * (t ** 0) * exp (-1.8 t)
+ (-200.911) * (t ** 0) * exp (-1.8 t)
+ (67.2981) * (t ** 0) * exp (-0.6 t)
+ (-5.71429) * (t ** 1) * exp (-0.6 t)
+ (-236.801) * (t ** 0) * exp (-1.2 t)
+ (1.94164) * (t ** 0) * exp (-1.3 t)
+ (190.687) * (t ** 0) * exp (-1 t)
+ (39.8632) * (t ** 0) * exp (-1.6 t)
+ (-34.149) * (t ** 0) * exp (-0.8 t)
+ (-22.2647) * (t ** 0) * exp (-0.5 t)
+ (48.6971) * (t ** 0) * exp (-2.1 t)
+ (75.1147) * (t ** 0) * exp (-2 t)
+ (0.506549) * (t ** 0) * exp (-3.5 t)
+ (1)
```

Table 2 - State Probability Expression For PERT Chain Absorbing State

```
+ (46.6468) * (t ** 0) * exp (-2.7 t)
+ (6.58467) * (t ** 1) * exp (-2.7 t)
+ (-13.5139) * (t ** 0) * exp (-3 t)
+ (1.2096) * (t ** 1) * exp (-3 t)
+ (853.74) * (t ** 0) * exp (-1.5 t)
+ (-923.911) * (t ** 0) * exp (-1.1 t)
+ (-1.11422) * (t ** 1) * exp (-1.1 t)
+ (1655.42) * (t ** 0) * exp (-1.7 t)
+ (-412.962) * (t ** 1) * exp (-1.7 t)
+ (1155.72) * (t ** 0) * exp (-1.8 t)
+ (279.099) * (t ** 1) * exp (-1.8 t)
+ (-2790.43) * (t ** 0) * exp (-1.8 t)
+ (5.27288) * (t ** 0) * exp (-0.6 t)
+ (-714.788) * (t ** 0) * exp (-1.2 t)
+ (236.801) * (t ** 1) * exp (-1.2 t)
+ (1828.72) * (t ** 0) * exp (-1.3 t)
+ (-3.20483) * (t ** 0) * exp (-1 t)
+ (-1355.52) * (t ** 0) * exp (-1.6 t)
+ (-17.1684) * (t ** 0) * exp (-0.8 t)
+ (1.04781) * (t ** 0) * exp (-0.5 t)
+ (26.3408) * (t ** 0) * exp (-2.1 t)
+ (242.907) * (t ** 0) * exp (-2 t)
+ (2.72702) * (t ** 0) * exp (-3.5 t)
+ (0)
```

Table 3 - Numerical Sensitivity Expression For the Absorbing State, Parameter μ_2

```
+ (26) * (t ** 0) * exp (-0.0003 t)
+ (0.00240006) * (t ** 1) * exp (-0.0003 t)
+ (9.0018e-08) * (t ** 2) * exp (-0.0003 t)
+ (5.994e-08) * (t ** 0) * exp (-1.0002 t)
+ (-1.80054e-11) * (t ** 1) * exp (-1.0002 t)
+ (-27) * (t ** 0) * exp (-0.0002 t)
+ (1)
```

Table 4a - Unreliability Expression for TMR System

```
+ (25.9892) * (t ** 0) * exp (-0.0003 t)
+ (0.00239898) * (t ** 1) * exp (-0.0003 t)
+ (8.9964e-08) * (t ** 2) * exp (-0.0003 t)
+ (-26.9892) * (t ** 0) * exp (-0.0002 t)
+ (1)
```

Table 4b - TMR Unreliability Expression Instantaneous Coverage Approximation

```
+ (26) * (t ** 0) * exp (-0.0003 t)
+ (0.0024) * (t ** 1) * exp (-0.0003 t)
+ (9e-08) * (t ** 2) * exp (-0.0003 t)
+ (-27) * (t ** 0) * exp (-0.0002 t)
+ (1)
```

Table 4c - TMR Unreliability Expression Perfect Coverage Approximation

+ (120004) * (t ** 0) * exp(-0.0003 t)
+ (10.0009) * (t ** 1) * exp(-0.0003 t)
+ (0.00030006) * (t ** 2) * exp(-0.0003 t)
+ (-0.00029964) * (t ** 0) * exp(-1.0002 t)
+ (8.9982e-08) * (t ** 1) * exp(-1.0002 t)
+ (-135000) * (t ** 0) * exp(-0.0002 t)
+ (14996) * (t ** 0) * exp(0 t)

Table 5 - Mean Up Time in [0,t] for TMR System

```
+ (-1.1995e+09) * (t ** 0) * exp (-0.0003 t)
+ (-359961) * (t ** 1) * exp (-0.0003 t)
+ (-27.0003) * (t ** 2) * exp (-0.0003 t)
+ (-0.00090018) * (t ** 3) * exp (-0.0003 t)
+ (-2.9946) * (t ** 0) * exp (-1.0002 t)
+ (0.00180018) * (t ** 1) * exp (-1.0002 t)
+ (1.34946e+09) * (t ** 0) * exp (-0.0002 t)
+ (270000) * (t ** 1) * exp (-0.0002 t)
+ (-1.4996e+08) * (t ** 0) * exp (0 t)
```

Table 6a - Numerical Sensitivity Expression for E[Y(t)] with respect to λ

```
+ (-3.99833) * (t ** 0) * exp (-0.0003 t)
+ (-0.00089992) * (t ** 1) * exp (-0.0003 t)
+ (-6.0018e-08) * (t ** 2) * exp (-0.0003 t)
+ (0.00059892) * (t ** 0) * exp (-1.0002 t)
+ (0.00029946) * (t ** 1) * exp (-1.0002 t)
+ (-8.9982e-08) * (t ** 2) * exp (-1.0002 t)
+ (-2.18834e-13) * (t ** 0) * exp (-0.0002 t)
+ (3.99773) * (t ** 0) * exp (0 t)
```

Table 6b - Numerical Sensitivity Expression for E[Y(t)] with respect to δ

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19. ABSTRACT (Continue on reverse if necessary and identify by block number):

Continuous time Markov chains are commonly used in system reliability modeling. Increasing system complexity, and non-Markovian behavior can drastically increase the size of a Markov model's state space. Special approximation techniques and numerical methods have been introduced to reduce the resources needed to solve Markov chain models. In this paper we discuss a method for automatically deriving exact transient solutions of Markov chains. The solutions derived are symbolic in t. Our approach can also provide solutions that are symbolic in other parameters. We extend our method to include parametric sensitivity analysis of the transient solution, and to provide cumulative measures of Markov chain behavior. We present three examples, one to show the use of our method in evaluating approximate solution techniques, one showing parametric sensitivity analysis of a large Markov model, and one demonstrating the computation of cumulative measures for an acyclic Markov reward processes.

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